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2D Thermocapillary Motion of Three Fluids in a Flat Channel

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Two-dimensional creeping motion of three immiscible, incompressible viscous fluids in a flat channel bounded by fixed solid walls, on which the temperature distribution is known, is investigated. The motion is induced only by the thermalcapillary forces beginning from the state of rest. Unsteady motion is described by finite analytic formulas obtained by Laplace transform in images. The evolution of the velocity fields to the stationary regime for specific liquids is obtained by the numerical inversion of Laplace transformation.

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It is well known that in a non-uniformly heated liquid a motion can arise. In some applications of liquid flows, a joint motion of two or more fluids with surfaces takes place. If the liquids are not soluble in each other, they form a more or less visual interfaces. The petroleum-water system is a typical example of this situation. At the present time modelling of multiphase flows taking into account different physical and chemical factors is needed for designing of cooling systems and power plants, in biomedicine, for studying the growth of crystals and films, in aerospace industry [1–4].

Nowadays, there are exact solutions of the Marangoni convection [5–7]. One of the first solutions was obtained in Napolitano [8]. This is the Poiseuille stationary flow of two immiscible liquids in an inclined channel. As a rule, all such flows were considered steady and unidirectional. The stability of such flows was investigated in [9,10]. As for non-stationary thermocapillary flows, studying of them began recently [11,12].

Thermocapillary convection problem for two incompressible liquids separated by a closed interface in a container was investigated in [13]. Local (in time) unique solvability of the problem was obtained in Holder classes of functions. The problem of thermalcapillary 3D motion of a drop was studied in [14]. Moreover, its unique solvability in Holder spaces with a power-like weight at infinity was established. Velocity vector field decreases at infinity in the same way as the initial data and mass forces, the temperature diverges to the constant which is the limit of the initial temperature at infinity.

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The present work is devoted to studying of solutions of a conjugate boundary value problem arising as a result of linearization of the Navier-Stokes system asupplemented with temperature equation. The description of the 2D creeping joint motion of three viscous heat conducting fluids in flat layers is also provided here. The motion arises due to thermocapillary forces imposed along two interfaces, after which the unsteady Marangoni convection begins. Such kind of convection can dominate in flows under microgravity conditions or in motions of thin liquid films.

1. Statement of problem

The two-dimensional motion of three viscous incompressible heat conducting fluids in the absence of mass forces is described by the system

$$
\begin{gathered}
u_t + uu_x + vu_y + \frac{1}{\rho}p_x = \nu(u_{xx} + u_{yy}),\\
v_t + uv_x + vv_y + \frac{1}{\rho}p_y = \nu(v_{xx} + v_{yy}),\\
u_x + v_y = 0,\\
\Theta_t + u\Theta_x + v\Theta_y = \chi(\Theta_{xx} + \Theta_{yy}).
\end{gathered} \tag{1}
$$

Here u, v are the components of the velocity vector, p is the pressure, Θ is the temperature, ρ is the density, ν is the kinematic viscosity, χ is the thermal diffusivity. The values of ρ, ν, χ are represented by constants.

We find the exact solution in the form

$$u(x, y, t), \ \ v(y, t), \ \ p(x, y, t), \ \ \Theta(x, y, t).$$

In such case, the first three equations of system (1) lead to the relations

$$
\begin{gathered}
u = w(y, t)x + u_1(y, t), \quad w + v_y = 0,\\
w_t + vw_y + w^2 = f(t) + \nu w_{yy},\\
\frac{1}{\rho}p = d(y, t) - \frac{f(t)}{2}x^2,\\
d_y = \nu v_{yy} - v_t - vv_y, \quad u_{1t} + vu_{1y} + u_1 w = 0
\end{gathered} \tag{2}
$$

with an arbitrary function $f(t)$.

With respect to the temperature field we assume that it has the form

$$\Theta = a(y, t)x^2 + a_1(y, t)x + b(y, t). \tag{3}$$

It will be seen, Eq. (3) has a good agreement with the conditions at the interfaces.

The stationary solution of the Navier-Stokes equations in the form (2) for $g = 0$ for pure viscous fluid was found for the first time by Hiemenz [15]. It describes the liquid inpingement from infinity on the plane $y = 0$ under the no slip condition on it. In the paper Brady and Acrivos [16], this solution for the flow between two plates or for the flow in a cylindrical tube (axisymmetric analogue of solution (2)) was applied.

It is known that the temperature dependence of the surface tension coefficient is the one of the most important factors leading to the dynamic variety of the interfacial surface. In the papers Bobkov and Gupalo [17], Gupalo and Ryazantsev [18] the stationary solutions in form (2), (3) was found at $a(y,t) \equiv 0$, $b = const$ for a flat layer with a free boundary $y = l = const$ and a solid wall $y = 0$. The non-uniqueness of solution depending on the physical parameters of the problem was revealed. A similar problem in the case of half space was investigated in Gupalo et al. [19].

Further, we assume that $u_1(y,t) \equiv 0, a_1(y,t) \equiv 0$. The latter means that the temperature field has an extremum at the point $x = 0$, more precisely, at $a(y,t) < 0$ it has a maximum and at $a(y,t) > 0$ it has a minimum. Let us apply the solution in the form (2), (3) to describe the joint flow of three immiscible liquids in flat layer $0 < y < l_3$ and take into account that the walls $y = 0$, $y = l_3$ are solid (see Fig. 1). By introducing the index $j = 1, 2, 3$ fixing the fluid and using

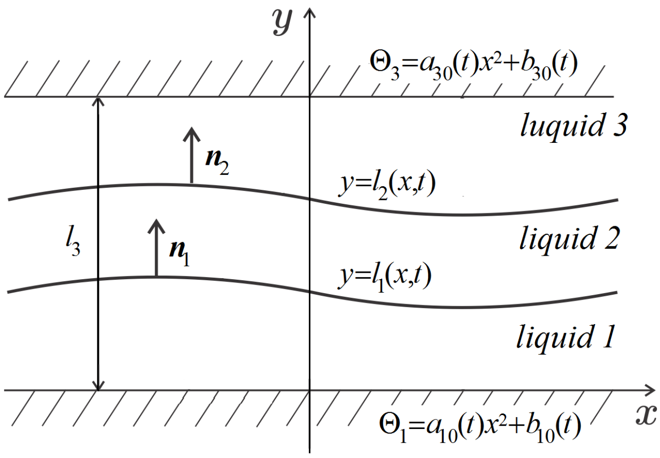

Fig. 1. The domain of the fluids flow

Eqs. (2), (3) we find that, in their domains, the unknowns satisfy the equations

$$
\begin{gathered}
w_{jt} + v_j w_{jy} + w_j^2 = \nu_j w_{jyy} + f_j(t), \\
v_{jt} + v_j v_{jy} + d_{jy} = \nu_j v_{jyy}, \\
w_j + v_{jy} = 0, \\
a_{jt} + 2w_j a_j + v_j a_{jy} = \chi_j a_{jyy}, \\
b_{jt} + v_j b_{jy} = \chi_j b_{jyy} + 2\chi_j a_j.
\end{gathered} \tag{4}
$$

On the interfaces $y = l_n(x,t)$, $n = 1, 2$, there are the conditions

$$
\begin{gathered}
w_1(l_1(x,t),t) = w_2(l_1(x,t),t), \quad w_2(l_2(x,t),t) = w_3(l_2(x,t),t), \\
v_1(l_1(x,t),t) = v_2(l_1(x,t),t), \quad v_2(l_2(x,t),t) = v_3(l_2(x,t),t),
\end{gathered} \tag{5}
$$

$$
l_{1t} + xw_1(l_1(x,t),t)l_{1x} = v_1(l_1(x,t),t), \quad l_{2t} + xw_2(l_2(x,t),t)l_{2x} = v_2(l_2(x,t),t), \tag{6}
$$

$$
\begin{gathered}
a_1(l_1(x,t),t) = a_2(l_1(x,t),t), \quad a_2(l_2(x,t),t) = a_3(l_2(x,t),t), \\
k_1 \frac{\partial a_1}{\partial n_1} = k_2 \frac{\partial a_2}{\partial n_1}, \quad k_2 \frac{\partial a_2}{\partial n_2} = k_3 \frac{\partial a_3}{\partial n_2},
\end{gathered} \tag{7}
$$

where $k_j > 0$ are the thermal conductivities, $\boldsymbol{n}_n = (-l_{nx}, 1)/\ (1 + l_{nx}^2)^{1/2}$ are normals to the curves $y = l_n(x, t)$, $n = 1, 2$. Dynamic conditions at the $y = l_n(x, t)$ are

$$
\begin{aligned}
&(p_1 - p_2)\mathbf{n}_1 + [2\rho_2\nu_2 D(\mathbf{u}_2) - 2\rho_1\nu_1 D(\mathbf{u}_1)]\mathbf{n}_1 = 2\sigma_1(\Theta_1)K_1\mathbf{n}_1 + \nabla_{11}\,\sigma_1,\\
&(p_2 - p_3)\mathbf{n_2} + [2\rho_3\nu_3 D(\mathbf{u}_3) - 2\rho_2\nu_2 D(\mathbf{u}_2)]\mathbf{n_2} = 2\sigma_2(\Theta_2)K_2\mathbf{n}_2 + \nabla_{11}\,\sigma_2.
\end{aligned}
\tag{8}
$$

In (8), D is the tensor of velocities deformation $\mathbf{u}_j = (xw_j(y, t),\ v_j(y, t))$ and, in the right-hand sides, $\nabla_{11} = \nabla - (\mathbf{n}\cdot\nabla)\mathbf{n}$ denotes the surface gradient, values $K_n = l_{nxx}(1 + l_{nx}^2)^{-3/2}$ are the average curvatures of the interfaces $y = l_n(x, t)$; $\sigma_1(\Theta_1)$, $\sigma_2(\Theta_2)$ are surface tension coefficients which depend on the temperature. For the majority of liquid media, the dependence $\sigma_n(\Theta_n)$ is well approximated by a linear one

$$
\sigma_n(\Theta_n) = \sigma_n^0 - æ_n\Theta_n,
\tag{9}
$$

where $æ_n > 0$ are the temperature coefficients of surfaces tension of lines $y = l_n(x, t)$. They are considered constants and determined experimentally.

Dynamic conditions (8) are given in the vector form. Projecting them to the tangential directions $\boldsymbol{\tau}_n = (1, l_{nx})/(1 + l_{nx}^2)^{1/2}$, using the dependence (9) and equalities (2) we obtain

$$
\begin{aligned}
&[\mu_2 D(\boldsymbol{u}_2) - \mu_1 D(\boldsymbol{u}_1)]\,\boldsymbol{n}_1\cdot\boldsymbol{\tau}_1 = -æ_1\nabla_{11}\Theta_1\cdot\boldsymbol{\tau}_1,\\
&[\mu_3 D(\boldsymbol{u}_3) - \mu_2 D(\boldsymbol{u}_2)]\,\boldsymbol{n}_2\cdot\boldsymbol{\tau}_2 = -æ_2\nabla_{11}\Theta_2\cdot\boldsymbol{\tau}_2,
\end{aligned}
\tag{10}
$$

where $\mu_j = \rho_j\nu_j$ are the dynamic viscosities of liquids. Projections (8) on the normals $\mathbf{n}_{1,2}$ provide the relations

$$
\begin{aligned}
&\rho_1 d_1(l_1(x, t), t) + \frac{[\rho_2 f_2(t) - \rho_1 f_1(t)]x^2}{2} - \rho_2 d_2(l_1(x, t), t) +\\
&\qquad\qquad + [2\mu_2 D(\boldsymbol{u}_2) - 2\mu_1 D(\boldsymbol{u}_1)]\boldsymbol{n}_1\cdot\boldsymbol{n}_1 =\\
&\quad = (\sigma_1^0 - æ_1[a_1(l_1(x, t), t)x^2 + b_1(l_1(x, t), t)])l_{1xx}(1 + l_{1x}^2)^{-3/2},\\
&\rho_2 d_2(l_2(x, t), t) + \frac{[\rho_3 f_3(t) - \rho_2 f_2(t)]x^2}{2} - \rho_3 d_3(l_2(x, t), t) +\\
&\qquad\qquad + [2\mu_3 D(\boldsymbol{u}_3) - 2\mu_2 D(\boldsymbol{u}_2)]\boldsymbol{n}_2\cdot\boldsymbol{n}_2 =\\
&\quad = \left(\sigma_2^0 - æ_2\left[a_2(l_2(x, t), t)x^2 + b_2(l_2(x, t), t)\right]\right) l_{2xx}(1 + l_{2x}^2)^{-3/2}.
\end{aligned}
\tag{11}
$$

Boundary conditions on the solid walls $y = 0, y = l_3$

$$
\begin{aligned}
&u_1(0, t) = 0,\ \ u_3(l_3, t) = 0,\ \ v_1(0, t) = 0,\ \ v_3(l_3, t) = 0,\\
&a_1(0, t) = a_{10}(t),\ \ a_3(l_3, t) = a_{30}(t)
\end{aligned}
\tag{12}
$$

with specified functions $a_{10}(t), a_{30}(t)$. Initial data for the velocities are zero (we study the properties of the solution of problem, which models the motion only under thermocapillarity forces)

$$
u_j(y, 0) = 0,\ \ v_j(y, 0) = 0,
\tag{13}
$$

moreover

$$
l_n(x, 0) = l_n^0(x),\ \ a_j(y, 0) = a_j^0(y).
\tag{14}
$$

Note the following features of the problem. It is nonlinear and inverse, since functions $f_j(t)$ are unknowns as well. It is easy to understand this, if we exclude the functions $v_j(y, t)$ from equations of mass conservation. Then the problem reduces to the conjugate problem for functions $w_j(y,t)$, $f_j(t)$, $a_j(y,t)$ и $l_n(x,t)$. The problem for functions $b_j(y,t)$ separates at the known functions $v_j(y,t)$ and $a_j(y,t)$, boundary conditions for the functions $b_j(y,t)$ are similar to conditions for the functions $a_j(y,t)$. The functions $d_j(y,t)$ can be restored by quadrature from the second equations (4) up to time functions. So the functions $w_j(y,t)$, $v_j(y,t)$ $a_j(y,t)$ are the solutions of nonlinear parabolic equations with boundary conditions (5)–(7), (12) and initial data (13), (14). The last two conditions in (5) and the fourth in (12) are helpful for determining of the functions $f_j(t)$.

We introduce the characteristic scales of length, time and functions w_j, v_j, a_j, d_j, f_j to simplify the problem (4)–(7), (10)–(14)

$$l_1^0,\ \frac{l_1^{02}}{\nu_1},\ \frac{æ_1 a^0 l_1^0}{\mu_1},\ \frac{æ_2 a^0 l_1^{02}}{\mu_1},\ a^0,\ \frac{æ_1 a^0 l_1^0}{\rho_1},\ \frac{æ_1 a^0}{\rho_1 l_1^0},$$

respectively, where $l_1^0 = const > 0$ is the average value of the layer thickness of the first liquid at $t = 0$, $a^0 = \max\limits_{t\geqslant 0} |a_{30}(t) - a_{10}(t)| > 0$ or $a^0 = \max\limits_j \max\limits_y |a_j^0(y)| > 0$ when $a_{10}(t) = a_{30}(t)$.

In the dimensionless variables, Marangoni number $M = æ_1 a^0 l_1^{03} \mu_1^{-1} \nu_1^{-1}$ appears at the nonlinear summands in eq. (4). The same will be in kinematic conditions (6) at the linear summands containing velocities. It is supposed that temperature coefficients of surface tension are comparable in magnitude $æ_1 \sim æ_2$ and $M \ll 1$. The latter takes place in thin layers or at very high viscosities. Thus, nonlinear summands can be neglected and equation becomes linear. In particular, kinematic conditions take the form $l_{nt} = 0$, so $l_n = l_n(x)$, $n = 1, 2$.

Let us turn to dynamic conditions (11). After introducing dimensionless variables, in the right side capillary number $Ca_n = a^0 l_1^{02} æ_1/\sigma_n^0$ appears instead of σ_1^0 и σ_2^0. In real conditions, for the majority of liquids $Ca_n \ll 1$, for example, for system water - air $Ca \sim 10^{-6}$. Therefore, at such capillary numbers, conditions (11) take the form $l_{nxx} = 0$ that is $l_n(x) = \alpha_n x + l_n^0$.

Further we assume that $\alpha_n = 0$ and the surfaces are planes $y = l_1^0$, $y = l_2^0 > l_1^0$, which are parallel to solid walls $y = 0, y = l_3$; index "0" in l_n^0 is omitted.

We can write out the whole linear problem in the dimensional form

$$w_{jt} = \nu_j w_{jyy} + f_j(t), \quad j = 1, 2, 3, \tag{15}$$

$$w_j(y, 0) = 0, \tag{16}$$

$$w_1(0, t) = 0, \ \ w_3(l_3, t) = 0, \tag{17}$$

$$w_1(l_1, t) = w_2(l_1, t), \ \ w_2(l_2, t) = w_3(l_2, t), \tag{18}$$

$$\begin{gathered} \mu_2 w_{2y}(l_1, t) - \mu_1 w_{1y}(l_1, t) = -2æ_1 a_1(l_1, t), \\ \mu_3 w_{3y}(l_2, t) - \mu_2 w_{2y}(l_2, t) = -2æ_2 a_2(l_2, t), \end{gathered} \tag{19}$$

$$\int_0^{l_1} w_1(y, t)\, dy = 0, \quad \int_{l_1}^{l_2} w_2(y, t)\, dy = 0, \quad \int_{l_2}^{l_3} w_3(y, t)\, dy = 0, \tag{20}$$

where $0 < y < l_1$ for $j = 1$, $l_1 < y < l_2$ for $j = 2$ and $l_2 < y < l_3$ for $j = 3$. Conditions (19) follow from (10), because $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = (1, 0)$ and $\nabla_{11}\Theta_{1,2} \cdot \boldsymbol{\tau}_{1,2} = 2a_{1,2}x$, according to the expression for the temperature (3). The first two equalities in (2) follow from kinematic conditions (6) and the last one is the no slip condition $v_3(l_3, t) = 0$.

We write the problem for the functions $a_j(y,t)$ as follows

$$a_{jt} = \chi_j a_{jyy}, \tag{21}$$

$$a_j(y,0) = a_j^0(y), \tag{22}$$

$$a_1(0,t) = a_{10}(t), \ \ a_3(l_3,t) = a_{30}(t), \tag{23}$$

$$a_1(l_1,t) = a_2(l_1,t), \ \ a_2(l_2,t) = a_3(l_2,t), \tag{24}$$

$$k_1 a_{1y}(l_1,t) = k_2 a_{2y}(l_1,t), \ \ k_2 a_{2y}(l_2,t) = k_3 a_{3y}(l_2,t). \tag{25}$$

2. Stationary flow

Problem (15)–(25) has a steady state, according to which the stationary flow of three immiscible, incompressible viscous fluids are described by formulas

$$\begin{gathered}
a_1^s(\xi) = \tilde{a}\xi + a_{10}^s, \quad 0 < \xi = y/l_1 < 1, \\
a_2^s(\xi) = \tilde{a}\left((\xi-1)\bar{k}_1 + 1\right) + a_{10}^s, \quad 1 < \xi < 1/\bar{l}_1, \\
a_3^s(\xi) = \bar{k}_1\bar{k}_2\tilde{a}(\xi - \frac{\bar{l}_2}{\bar{l}_1}) + a_{30}^s, \quad 1/\bar{l}_1 < \xi < \bar{l}_2/\bar{l}_1, \\
\bar{w}_1^s(\xi) = \frac{\bar{l}_1(\bar{l}_1-1)}{m_1}\left(2\xi - 3\xi^2\right)\left[M_1 - \frac{m_5}{m_4}(\bar{l}_1-1)\right], \quad 0 < \xi = y/l_1 < 1, \\
\bar{w}_2^s(\xi) = -\left[\frac{3\bar{l}_1 m_5}{m_4}(\xi-1)^2 + \frac{\bar{l}_1(\bar{l}_1-1)(1+4\bar{\mu}_1(\xi-1))}{m_1}\left(M_1 - \frac{m_5(\bar{l}_1-1)}{m_4}\right) + 2M_1(\xi-1)\right], \\
1 < \xi < 1/\bar{l}_1, \\
\bar{w}_3^s(\xi) = 2\bar{\mu}_2\left(\xi - \frac{\bar{l}_2}{\bar{l}_1}\right)\left[-\frac{3m_6\bar{l}_1}{4(\bar{l}_2-1)}(\xi + \frac{\bar{l}_2-2}{\bar{l}_1}) + \frac{m_2 m_5(\bar{l}_1-1)}{m_1 m_4} - \frac{M_1\bar{l}_1^2}{m_1} - M_2\right], \\
1/\bar{l}_1 < \xi < \bar{l}_2/\bar{l}_1, \\
f_1^s = \frac{6\bar{l}_(\bar{l}_1-1)}{m_1}\left(M_1 - \tfrac{m_5}{m_4}(\bar{l}_1-1)\right), \quad f_2^s = \frac{6\bar{l}_1 m_5}{\bar{\nu}_1 m_4}, \quad f_3^s = \frac{3\bar{l}_1\bar{\nu}_2\bar{\mu}_2 m_6}{\bar{\nu}_1(\bar{l}_2-1)},
\end{gathered} \tag{26}$$

where $\bar{w}_j^s = w_j^s\nu_1^{-1}\bar{l}_1^2$, $\tilde{a} = \bar{l}_1(a_{30}^s - a_{10}^s)/m$, functions a_j^s, w_j^s, f_3^s are the stationary solutions of problems (21)–(25), (15)–(20), a_{10}^s, a_{30}^s are the constant values on the walls $y = 0$ and $y = l_3$ respectively, $\bar{l}_1 = l_1/l_2$, $\bar{l}_2 = l_3/l_2$, $\bar{k}_n = k_n/k_{n+1}$, $\bar{\mu}_n = \mu_n/\mu_{n+1}$, $\bar{\nu}_1 = \nu_1/\nu_2$, $\bar{\nu}_2 = \nu_3/\nu_2$, $M_n = æ_n a_n^s(l_n)\nu_1^{-1}\mu_2^{-1}$ are the Marangoni numbers and constants m, m_1, m_2, m_3, m_4, m_5, m_6 are calculated by the formulas

$$m = \bar{l}_1 + (1-\bar{l}_1)\bar{k}_1 + (\bar{l}_2-1)\bar{k}_1\bar{k}_2, \quad m_1 = \bar{l}_1^2 - 2\bar{\mu}_1\bar{l}_1(\bar{l}_1-1),$$

$$m_2 = 3\bar{l}_1^2 - 4\bar{\mu}_1\bar{l}_1(\bar{l}_1-1), \quad m_3 = \bar{l}_1^2 - \bar{\mu}_1\bar{l}_1(\bar{l}_1-1),$$

$$m_4 = m_2\bar{\mu}_2(\bar{l}_2-1) - 4m_3(\bar{l}_1-1),$$

$$m_5 = \frac{\bar{\mu}_2(\bar{l}_2-1)(M_1\bar{l}_1^2 + M_2 m_1)}{\bar{l}_1-1} - 2M_1\bar{l}_1^2,$$

$$m_6 = \frac{m_2 m_5}{m_1 m_4}(\bar{l}_1-1) - \frac{M_1\bar{l}_1^2}{m_1} - M_2.$$

3. Non-stationary motion

To describe the non-stationary motion of three viscous thermally conducting liquids, the Laplace transformation is applied to problems (15)–(20), (21)–(25) (assuming that initial data (22) are zero). As a result, we come to boundary value problem for images $\hat{a}_j(y,p)$ of functions $a_j(y,t)$

$$p\chi_j^{-1}\hat{a}_j(y,p) - \hat{a}_{jyy}(y,p) = 0, \tag{27}$$

$$\hat{a}_1(0,p) = \hat{a}_{10}(p),\ \ \hat{a}_3(l_3,p) = \hat{a}_{30}(p), \tag{28}$$

$$\hat{a}_1(l_1,p) = \hat{a}_2(l_1,p),\ \ \hat{a}_2(l_2,p) = \hat{a}_3(l_2,p), \tag{29}$$

$$k_1\hat{a}_{1y}(l_1,p) = k_2\hat{a}_{2y}(l_1,p),\ \ k_2\hat{a}_{2y}(l_2,p) = k_3\hat{a}_{3y}(l_2,p) \tag{30}$$

and images $\hat{w}(y,p)$ of functions $w(y,t)$

$$p\nu_j^{-1}\hat{w}_j(y,p) - \hat{w}_{jyy}(y,p) = \nu_j^{-1}\hat{f}_j(p), \tag{31}$$

$$\hat{w}_1(0,p) = 0,\ \ \hat{w}_3(l_3,p) = 0, \tag{32}$$

$$\hat{w}_1(l_1,p) = \hat{w}_2(l_1,p),\ \ \hat{w}_2(l_2,p) = \hat{w}_3(l_2,p), \tag{33}$$

$$\begin{aligned} \mu_2\hat{w}_{2y}(l_1,p) - \mu_1\hat{w}_{1y}(l_1,p) &= -2æ_1\hat{a}_1(l_1,p), \\ \mu_3\hat{w}_{3y}(l_2,p) - \mu_2\hat{w}_{2y}(l_2,p) &= -2æ_2\hat{a}_2(l_2,p), \end{aligned} \tag{34}$$

$$\int_0^{l_1}\hat{w}_1(y,p)\,dy = 0,\ \ \int_{l_1}^{l_2}\hat{w}_2(y,p)\,dy = 0,\ \ \int_{l_2}^{l_3}\hat{w}_3(y,p)\,dy = 0. \tag{35}$$

In condition (28) and equation (31), $\hat{a}_{10}(p), \hat{a}_{30}(p), \hat{f}_j(p)$ are images of functions $a_{10}(t), a_{30}(t), f(t)$ respectively.

The solutions of problems (27)–(30), (31)–(35) can be written as

$$\begin{aligned} \hat{a}_j(y,p) &= d_j^1 \operatorname{sh}\sqrt{\frac{p}{\chi_j}}y + d_j^2 \operatorname{ch}\sqrt{\frac{p}{\chi_j}}y, \\ \hat{w}_j(y,p) &= c_j^1 \operatorname{sh}\sqrt{\frac{p}{\nu_j}}y + c_j^2 \operatorname{ch}\sqrt{\frac{p}{\nu_j}}y + \frac{\hat{f}_j(p)}{p}. \end{aligned} \tag{36}$$

The values d_j^1, d_j^2, c_j^1, c_j^2 and $\hat{f}_j(p)$ are determined from the boundary conditions (28)–(30), (32)–(35). The type of these values is not presented here because of its complexity.

Remark 1. *The solution for the functions $a_j(y,t)$ was obtained for the zero initial condition (22). Since the problem (21)–(25) is linear, this problem can be solved for non-zero conditions and when conditions (23) are uniform, boundary conditions (24), (25) remain unchanged and equation (21) has the form $p\hat{a}_j(y,p) - \chi_j^{-1}\hat{a}_{jyy}(y,p) - a_j^0(y) = 0$.*

Let us assume that $\lim\limits_{t\to\infty} a_{k0}(t) = a_{k0}^s, k = 1,3$. Using formulas (36) and expressions for the values $d_j^1, d_j^2, c_j^1, c_j^2$, $\hat{f}_j(p)$ we can prove the limit equalities

$$\begin{gathered} \lim_{p\to 0} p\hat{a}_j(y,p) = a_j^s(y),\ \ \lim_{p\to 0} p\hat{w}_j(y,p) = w_j^s(y), \\ \lim_{p\to 0} p\hat{f}_j(p) = f_j^s, \end{gathered} \tag{37}$$

where $a_j^s(y), w_j^s(y), f_j^s$ is determined by formulas (26).

4. Numerical results

Let us apply the numerical method of the inverse Laplace transformation to obtained formulas (36). It is enough to show only the pictures for velocities, because they have zeal physical inrepretation. All numerical calculations were made for the system of liquid silicon - water - air (thickness of the layers is the same and equal to 1 mm). The corresponding values of the defining parameters are given in Tab. 1.

Table 1. Physical properties of liquids

Item	liquid silicon	water	air
ρ, kg/m^3	956	998	1.205
$\nu \times 10^{-6}$, m$^2/s$	10.2	1.004	15.11
k, kg $\cdot$ m$/s^3 \cdot K$	0.133	0.597	0.00257
$\chi \times 10^{-6}$, m$^2/s$	0.0675	0.143	21
æ $\times 10^{-5}$, kg$/s^2 \cdot K$	6.4	15.14	—

Figs. 2, 3 show the profiles of the dimensionless functions $\bar{w}_j^s(\xi)$ (see 26) and transverse velocity $\bar{v}_j^s(\xi)$ for the case when $M_1 = M_2 = 0.0005$. Expressions for the velocities $\bar{v}_j^s(\xi) = v_j^s \nu_1^{-1} l_1$ were found from the equation of mass conservation (the third equation in (4))

$$v_1(y,t) = -\int_0^y w_1(y,t)\,dy, \quad v_2(y,t) = -\int_{l_1}^y w_2(y,t)\,dy,$$
$$v_3(y,t) = -\int_{l_2}^y w_3(y,t)\,dy.$$

In particular, Fig. 2 shows that the function $\bar{w}_j^s(\xi)$ is negative close to the interfaces $\xi = 1$ and $\xi = \bar{l}_1^{-1} = 2$, so the reverse flow arises here.

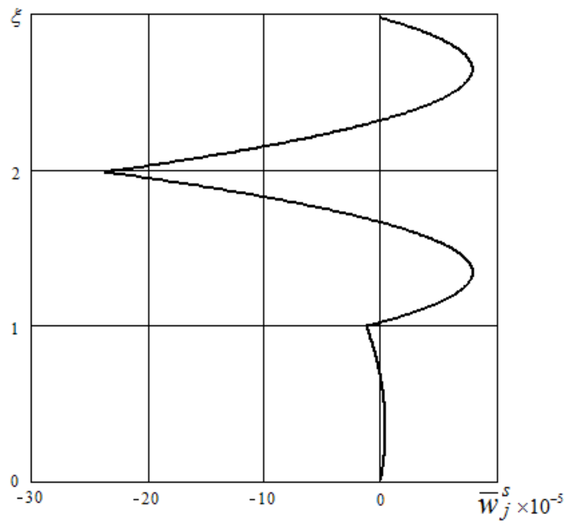

Fig. 2. The stationary profile of function $\bar{w}_j^s(\xi)$

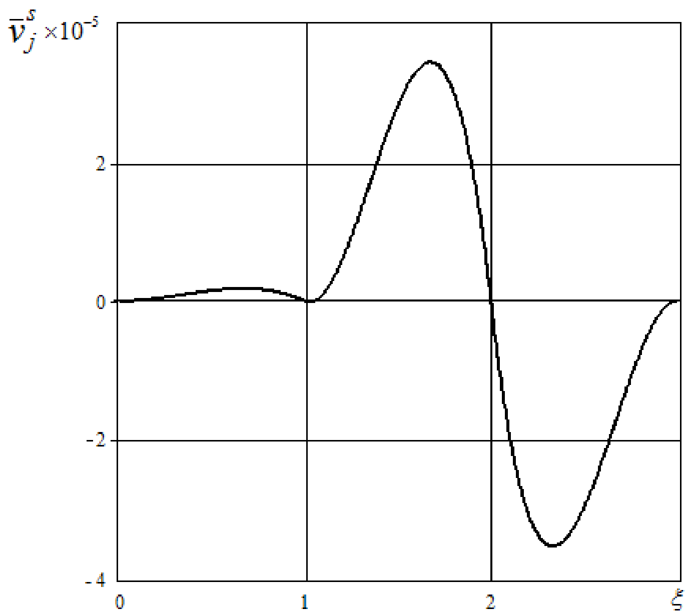

Fig. 3. Stationary profile of transverse velocity $\bar{v}_j^s(\xi)$

Fig. 4 shows the velocity field in the layers and Fig. 5 shows the velocity field close of the interfaces $\xi = 1$ and $\xi = 2$. Since $\bar{a}_1^s(1)$, $\bar{a}_2(\bar{l}_1^{-1}) > 0$ ($\bar{a}_1^s(1) = 79.7$, $\bar{a}_2(\bar{l}_1^{-1}) = 33.1$), the temperature field at $x = 0$ has a minimum (see (3)). Consequently, in the axial x direction temperature increases and the surface tension decreases (see (9)), so reverse flow occurs close to the interfaces.

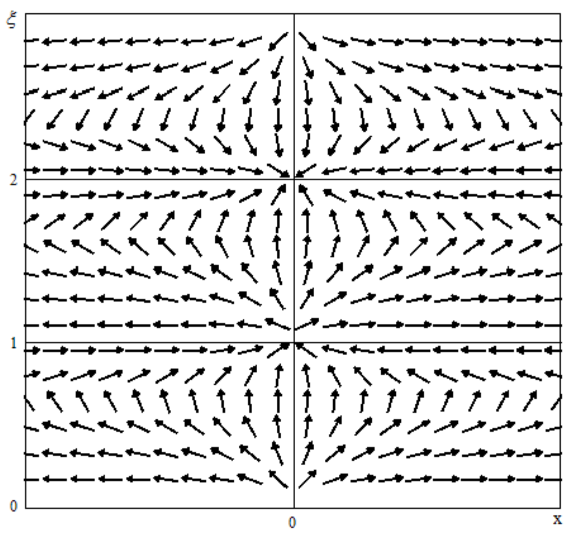

Fig. 4. The velocities field

Fig. 6 shows the convergence evolution of functions $\bar{w}_j(\xi,\tau)$ and transverse velocity $\bar{v}_j(\xi,\tau)$ to the stationary regime for the case when $\bar{a}_1(0,\tau) = 1 + 5e^{-0.05\tau}\sin(\tau)$, $\bar{a}_3(3,\tau) = 0$, $\tau = \nu_1 l_1^{-1} t$

is the dimensionless time (in our case, the real time $t = 10^2\tau$). As it is seen in the pictures, at $\tau = 800$ curves almost coincide with the stationary solution. Fig. 7 shows the results of calculations when $\bar{a}_1(0,\tau) = 2\sin(0.01\tau)$, $\bar{a}_3(3,\tau) = 0$. That is the limit of $\bar{a}_1(0,\tau)$ at $\tau \to \infty$ does not exist and, as it can be seen from the figure, the velocity field does not converge to the stationary one.

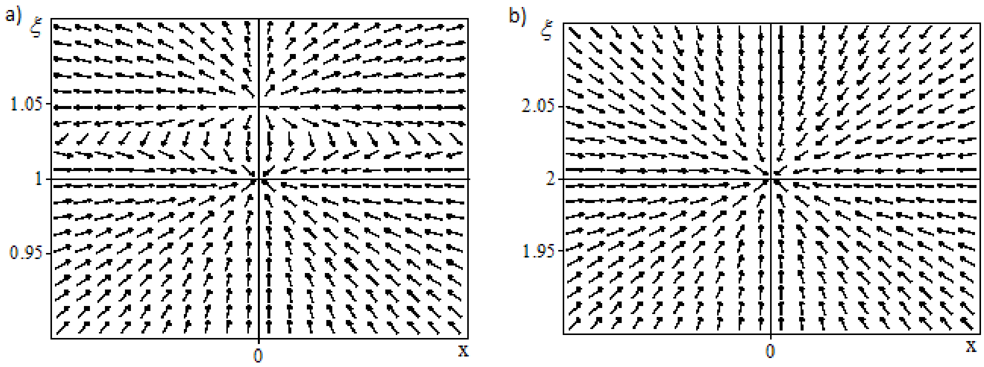

Fig. 5. The velocities field close to the interfaces $\xi = 1(a)$ и $\xi = \bar{l}_1^{-1} = 2(b)$

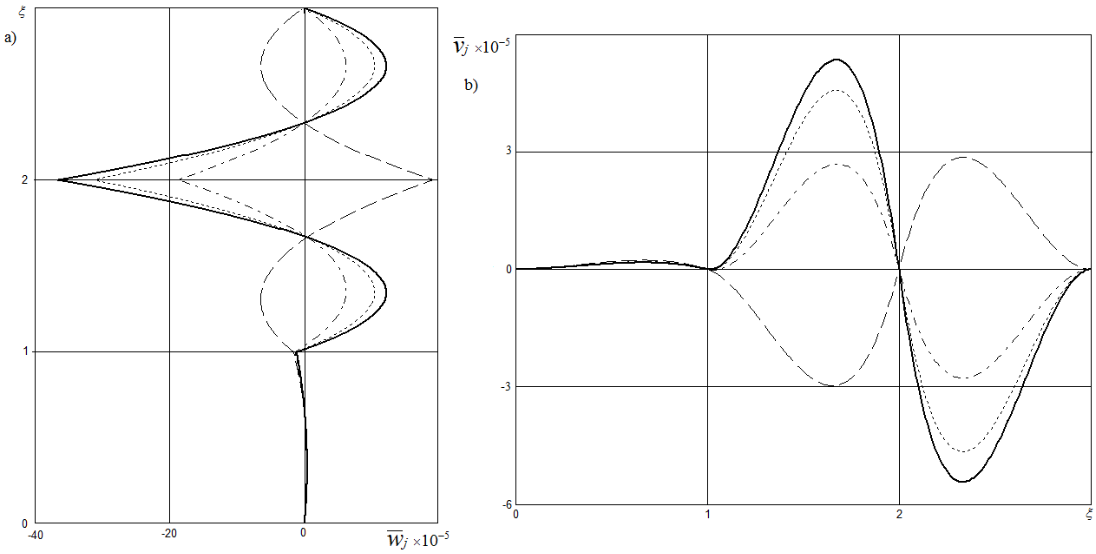

Fig. 6. The convergence evolution of functions $\bar{w}_j(\xi,\tau)$ (a) and transverse velocity $\bar{v}_j(\xi,\tau)$ (b) to the stationary regime: $\bar{a}_1(1,\tau) = 1 + 5e^{-0.05\tau}\sin(\tau)$, $\bar{a}_3(3,\tau) = 0$, stationary regime (—), $\tau = 24\,(--)$, $\tau = 400\,(-\cdot-)$, $\tau = 800\,(\cdots)$

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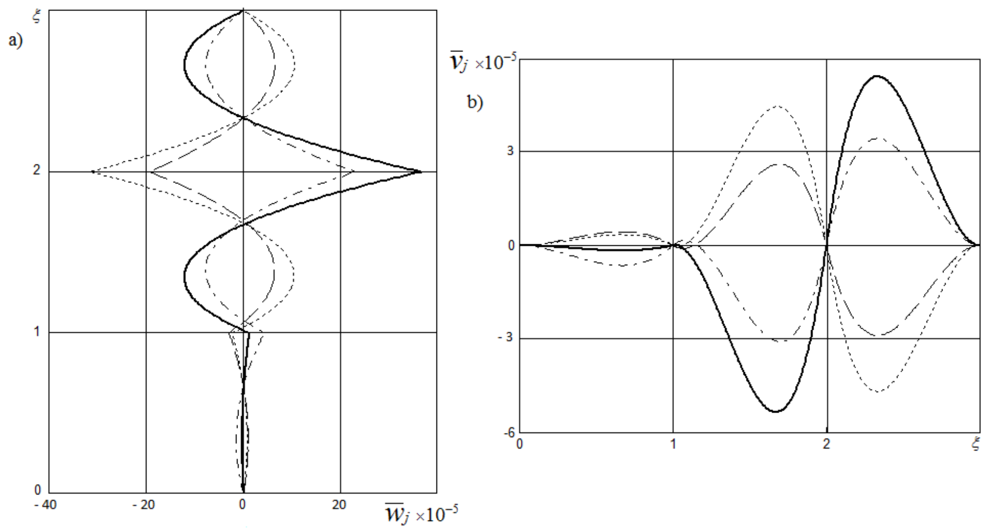

Fig. 7. The dimensionless profiles of functions $\bar{w}_j(\xi,\tau)$ (a) and transverse velocity $\bar{v}_j(\xi,\tau)$ (b): $\bar{a}_1(1,\tau) = 2\sin(0,01\tau)$, $\bar{a}_3(3,\tau) = 0$, stationary regime (—), $\tau = 470\,(--)$, $\tau = 850\,(-\cdot-)$, $\tau = 1070\,(\cdots)$

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Двумерное термокапиллярное движение трех жидкостей в плоском канале

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Исследовано двумерное ползущее движение трех несмешивающихся несжимаемых вязких теплопроводных жидкостей в плоском канале, ограниченном твердыми неподвижными стенками, на которых известно распределение температур. В образах по Лапласу построено точное нестационарное решение в виде квадратур и приведены некоторые численные результаты поведения полей скоростей и температур в слоях.

Ключевые слова: термокапиллярность, поверхность раздела, математическое моделирование, численные эксперименты.